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Electrons and vortex lines in He II, II. Theoretical analysis of capture and release experiments†

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Abstract. The theory of capture and release of an electron bubble from a quantized vortex line in rotating He II has been presented in part I, along with a derivation of the kinetic coefficients for capture and release to lowest order in the external field strength. In the present communication we use this information to analyse the available experimental data on capture and release, and show that the field dependence of both the cross section and lifetime is accurately predicted by the theory of Brownian motion of an ion in the presence of a static vortex potential, as well as is the temperature dependence of the cross section for $T \gtrsim 1{\cdot}4$ K.

1. Introduction

In part I (McCauley and Onsager 1975) we constructed the theoretical apparatus necessary to discuss the processes of capture and release of a negative ion (electron-bubble of radius R) from a static vortex line in He II (circulation $\kappa = h/m$, velocity field $v_s = \kappa/2\pi r$ where r is the ion–vortex separation), this theory being based upon the Smoluchowski equation for the probability density to find an ion–vortex separation $\boldsymbol{r}$ at time t, the ion having a potential energy $\phi = \phi_v + \phi_\epsilon$. $\phi_\epsilon = -eEx$ is the potential energy due to the external field (E in V cm^{-1}) and $\phi_v \simeq 2CkT/r^2$ is the ion–vortex potential energy (valid for $r/R \gg 1$) with force constant

$$2CkT = \frac{3}{2}\frac{\rho_s}{2}\left(\frac{\hbar}{m}\right)^2\frac{4}{3}\pi R^3.$$

The capture of an ion by a vortex line is described by the kinetic equation

$$-\mathrm{d}n_i/\mathrm{d}t = \mathrm{d}\nu/\mathrm{d}t = An_in_v - KA\nu \tag{1}$$

where n_i, n_v and ν are the average densities of free ions, vortex lines and captured ions respectively, and the rate coefficients A and KA are obtained from the Smoluchowksi equation (I). By an appropriate choice of two sets of boundary conditions we solved the Smoluchowski equation via a perturbation approach and obtained, to lowest order in the external field strength, the following results.

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(i) With

$$b_0 = \tfrac{3}{2}/(0{\cdot}577 - \ln \beta q) \tag{2}$$

and $\beta q = (eE/kT)\sqrt{C}$ we have a rate coefficient for capture given by

$$A = 2\pi\omega kTb_0, \tag{3}$$

where ω is the mobility of the ion through the normal fluid (primarily rotons in the temperature range 1–2 K). For $R = 20$ Å, $\beta q = 1{\cdot}74 \times 10^{-2}\,(\rho_s/T^3)^{1/2}E$ and the 'weak-field' approximation is defined by $\beta q \ll 1$ (near 1·4 K this requires $E \lesssim 70$ V cm^{-1}). The capture cross section is then given by

$$\sigma = \pi b_0/\beta. \tag{4}$$

(ii) For $\beta q \ll 1$ we have shown that the escape rate ($P = KA$) is given by

$$P \simeq A/\nu(0), \tag{5}$$

where $\nu(0) = \int_{r \lesssim 2\sqrt{C}} e^{-u_v}\, d^2r$ is the bound-state partition function and u_v is the exact ion–vortex potential energy divided by kT (I). An estimate for $\nu(0)$ is given by McCauley (1974) where it is shown that the well-known discrepancy in the temperature dependence of the trapping lifetime results, at least in large part, from the neglect of thermal fluctuations in the position of the vortex line. Here we are concerned only with the external electric field dependence of the trapping lifetime, and this is given by

$$\tau = \frac{1}{P} \propto \frac{1}{b_0} \tag{6}$$

for $\beta q \ll 1$.

Note that for $\beta q \ll 1$, τ^{-1} and $E\sigma$ have the same field dependence. This fact will be used to discuss the validity of two contradictory sets of capture cross section data.

We remind the reader that in I we showed that the trapping lifetime given by Donnelly and Roberts (1969) can be valid only as an asymptotic approximation in the limit $E \to \infty$. It is the purpose of the present communication to show that the available experimental data on capture and release of negative ions from vortex lines are described by the *weak-field* limit presented above (ie the data correspond to the approximation defined by the limit $E \to 0$).

2. Tanner's cross section measurements

Tanner (1966) has measured the capture cross section for a wide range of temperature and external field values. We consider first the case $T \lesssim 1{\cdot}55$ K, since in this case escape is negligible for all times of experimental interest. We must first generalize equation (1) to account for diffusion of the free ions while passing from source to collector. With $An_in_v = J_rn_v\sigma$, where $\boldsymbol{J} = e^{-u}\nabla n_i\, e^u$ and u is the external field potential, the correct generalization is

$$\partial n_i/\partial t = D\nabla \,.\, e^{-u}\nabla n_i\, e^u - J_rn_v\sigma + P\nu \tag{7}$$

$$\partial \nu/\partial t = J_rn_v\sigma - P\nu, \tag{8}$$

which gives (with I as the total free particle current)

$$dI/I = -n_v\sigma\, dx \tag{9}$$

whenever escape is negligible ($Pv = 0$), the flow is steady and the external field is uniform (Tanner's geometry was rectangular, so E is roughly a constant if the space charge effects can be neglected). This gives

$$\ln(I_F/I_0) = -n_v\sigma L \tag{10}$$

where L is the distance from ion source to collector, I_0 is the current from the source and I_F is the current at the collector. The comparison with experiment is given in figure 1. If $R \sim 16$–17 Å, for which there is evidence from Stokes radius measurements in He I well above the lambda point (Ahlers and Gamota 1972), then the theoretical cross sections will be slightly reduced.

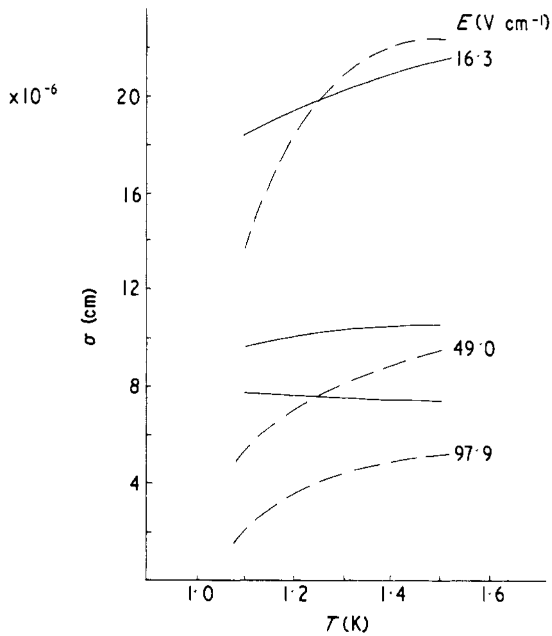

Figure 1. σ–T (K) curves for $E = 16{\cdot}3$, $49{\cdot}0$ and $97{\cdot}9$ V cm^{-1}. The dashed curves represent Tanner's (1966) figure 7 while our theoretical prediction is given by the solid curves. (Note that our lowest-order approximation in the field strength clearly fails for the case of $E = 97{\cdot}9$ V cm^{-1}. This is in accordance with the accuracy of the lowest-order approximations to the modified Bessel functions I_μ and K_μ (McCauley 1972).)

While it is clear that we have a good description of the field dependence (with the exception of the case $E = 97{\cdot}7$ V cm^{-1} due to the fact that our lowest-order approximation in field strength is valid only for $E < 70$ V cm^{-1}), there is an apparent discrepancy in the predicted temperature dependence when $T \lesssim 1{\cdot}4$ K. This latter conclusion is also supported by Douglass' data (see figure 2).

3. Douglass' capture rate measurements

Douglass (1966) has measured a quantity closely related to the capture rate for $T \sim 1{\cdot}20$–$1{\cdot}65$ K and $E \sim 5$–100 V cm^{-1}. His experimental apparatus consisted of two concentric

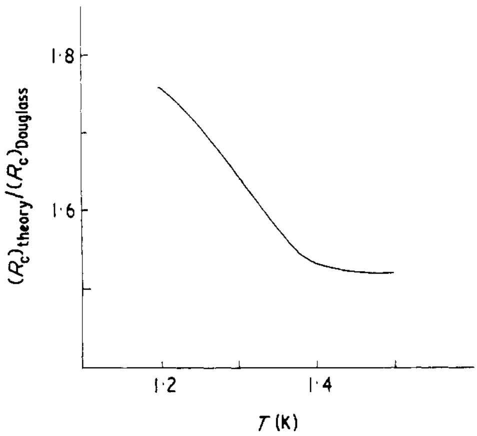

Figure 2. Plot of $(R_c)_{\text{theor}}/(R_c)_{\text{observed}}$ indicating a possible discrepancy between theory and experiment for $T \lesssim 1{\cdot}4$ K, a conclusion also suggested by figure 1.

cylinders with an ion source on an outer cylinder of radius b, a collector on an inner cylinder of radius a and He II in the space between the two cylinders. In this case the external field is non-uniform $E \propto 1/r$ where we take the origin to be the centre of the two concentric cylinders) so that $I = \oint J_r r\, \mathrm{d}\theta$ is the total current into a circle of radius r and u is the potential of the external field divided by $kT\,(u \propto \ln r)$. As before, we may set $Pv \simeq 0$ for $T \lesssim 1{\cdot}55$ K and as Douglass' (1966) figure 2 indicates the existence of steady flow from source to collector after less than one second, the approximation of interest to us is

$$\nabla \,.\, J - J_r n_v \sigma \simeq 0 \tag{11}$$

or

$$\ln(I_0/I_F) = n_v \int_a^b \sigma \,\mathrm{d}r \tag{12}$$

where I_0 is the current at $r = b$ and I_F the current at $r = a$ (the charge per second trapped by the vortex lattice is therefore $I_T = I_0 - I_F$).

In an attempt to approximate equation (12), Douglass assumed

$$E\sigma = C_1 E + C_2 \tag{13}$$

(C_1, C_2 constant) and defined an average field by the formula

$$E_{av}\sigma(E_{av}) = \frac{1}{b-a}\int_a^b \sigma \,\mathrm{d}r$$

(the result is $E_{av} = 2\Delta V/[(b+a)\ln(b/a)]$ where ΔV is the potential difference between the two cylinders). That (13) is not legitimate is clear from equations (2) and (4). However, we can easily make contact with Douglass' data because he really plots the quantity

$$\frac{1}{n_v}\frac{1}{(b-a)}\ln(1+I_T/I_F),$$

referred to by Douglass as '$E_{av}\sigma(E_{av})$' but hereafter denoted by R_c. As a function of Douglass' average field we have†

$$R_c = \frac{1}{b-a}\int_a^b \sigma\,\mathrm{d}r = \frac{3\pi kT}{e}\xi^2\,\mathrm{e}^{-1\cdot 15}E_{av}^2\Delta E_i \tag{14}$$

where (with $R \sim 20$ Å) $\xi^2 = (\rho_s/T^3)(1{\cdot}74\times 10^{-2})^2$, $\Delta E_i = E_i(2Z_b) - E_i(2Z_a)$,

$$E_i(z) = \int_{-\infty}^{Z} (\mathrm{e}^t/t)\,\mathrm{d}t$$

(Abramowitz and Stegun 1969)

$$Z_a = 3{\cdot}626 - \tfrac{1}{2}\ln(\rho_s/T^3) - \ln E_{av}$$

and $Z_b = 5{\cdot}134 - \frac{1}{2}\ln(\rho_s/T^3) - \ln E_{av}$. This gives

$$R_c = \tfrac{1}{2}\times 10^{-7}(\rho_s/T^2)E_{av}^2\Delta E_i. \tag{15}$$

Note that R_c is clearly a measure of the capture rate since $A \propto E\sigma$, $E \propto 1/r$ and $R_c \propto \int_a^b \sigma\,\mathrm{d}r$. Again, the theoretical temperature dependence is correct only for $T \gtrsim 1{\cdot}4$ K (see figure 2) as was suggested by Tanner's data. However, from figure 3 we see that

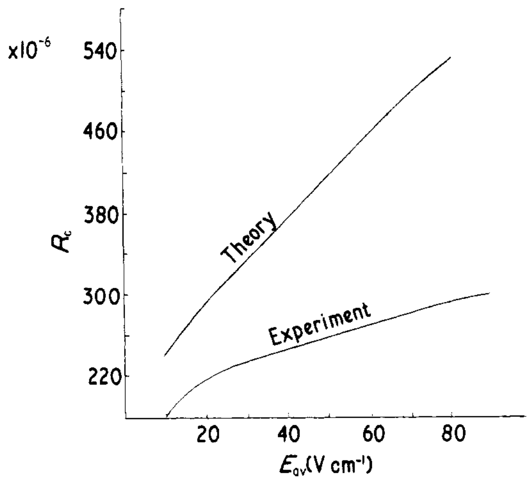

Figure 3. $R_c \times 10^6$ plotted against E_{av}(V cm^{-1}) for $T = 1{\cdot}4$ K, the theoretical prediction being given by the upper curve and Douglass' results by the lower curve (our lowest-order approximation in the field strength fails for $E_{av} \gtrsim 70$ V cm^{-1} at this temperature).

Douglass' R_c is too small and increases much too slowly with E_{av}. There are two independent experiments favouring our theoretical result—Tanner's cross section data and Pratt's data on the trapping lifetime. Recalling that for weak fields τ and $(E\sigma)^{-1}$ have the same E dependence, we state here (as will be seen in § 4) that the present theory is in close agreement with Pratt's data on escape. Therefore, the conclusion is drawn that Douglass' data are in error. The reason for the discrepancy is unclear, but the following observation may be relevant: if one calculates the predicted rate of capture per vortex line (given approximately by An_i) under Douglass' experimental conditions ($I_0 \sim 10^{-12}$ A), one finds a capture rate per vortex line of the order of 10^2 electrons per second.

† The correct values are $a = 0{\cdot}76$ cm, $b = 3{\cdot}43$ cm (Douglass, private communication).

Since the line is roughly 2 cm long one quickly sees that the present formula for A may not be applicable to Douglass' results, since the Coulomb repulsion between trapped and incoming charges may not be negligible. This amounts to suggesting that the slope in Douglass' figure 2 may be linear only for $\tau \gtrsim 5$ sec and should not be extrapolated to the origin as he has done. Douglass (1966) has stated that his data are in agreement with an early theoretical result of Donnelly's. However, as first noted by Tanner (1966), Douglass was forced to assume an unrealistically small ion–vortex force constant (corresponding to a bubble radius $R \simeq 5$–6 Å) in order to obtain this agreement, so it cannot be considered significant.

When escape cannot be neglected during times of experimental interest, what is measured is not the cross section itself but rather $\langle \sigma\, \mathrm{e}^{-Pt} \rangle$, where t is the transit time of a trapped ion along the vortex line from the scattering region to a collector perpendicular to the axis of rotation whenever the captured ions are continually collected throughout the experiment (Donnelly 1969). A suitable average for e^{-Pt} will be suggested by the experimental arrangement. For example, if $E_{\|}$ and $\mu_{\|}$ are the external field and mobility along the vortex lines, then (with $v_{\|} = \mu_{\|} E_{\|}$) a suitable average will be given by

$$\langle \mathrm{e}^{-Pt} \rangle = (1/a) \int_{L-a}^{L} \mathrm{e}^{-PZ/v_{\|}}\, \mathrm{d}Z = \mathrm{e}^{-PL/v_{\|}} (v_{\|}/Pa)(\mathrm{e}^{Pa/v_{\|}} - 1) \tag{16}$$

if L is equivalent to the maximum distance from the scattering region to the collector and a is equivalent to the size of the scattering region (a is roughly equal to the size of the source since the diffusion is negligible during most times of experimental interest). This reduces to $\mathrm{e}^{-PL/v_{\|}}$ whenever $Pa/v_{\|} \ll 1$.

In the event that the captured ions are not continually flushed along the lines and collected throughout the experiment, then $\sigma\, \mathrm{e}^{-Pt}$ is the 'measured cross section' and t is just the length of the experiment.

4. Pratt's trapping lifetime measurements

Pratt (1967) and Pratt and Zimmerman (1969) have presented the results of trapping lifetime measurements for $E \sim 1$–$80\ \mathrm{V\,cm^{-1}}$ and $T \sim 1{\cdot}6$–$1{\cdot}7$ K. In this discussion we will consider only the field dependence of τ without regard for its absolute magnitude, since the temperature dependence predicted by assuming a classical, static vortex line is grossly incorrect (McCauley 1974, Pratt 1967).

Let us consider Pratt's measuring procedure. After setting up a steady flow from source to collector (source on inner cylinder of radius a and $E \propto 1/r$ with origin of coordinates at the centre of the two cylinders), he shut off the free ion source and then observed the amount of charge still remaining trapped after a time interval t, assuming a charge decay law $Q = Q_0\, \mathrm{e}^{-Pt}$. We will see that (due to the cylindrical geometry) this will be true only if the average field E_{av} (Pratt and Zimmerman 1969) is less than a value determined by the dimensions of the experimental apparatus. Without regard for the actual magnitude of τ, we will investigate the consequences of

$$\tau \propto \tau' = 0{\cdot}577 - \ln \beta q \tag{17}$$

since $\tau \propto b_0^{-1}$ (see equation (6)). Since $E \propto 1/r$ we have $\beta q \propto \Delta V/r$ where ΔV is the potential difference between the two cylinders. For $t > 0$ (free ion source off),

$$\mathrm{d}\nu/\mathrm{d}t = -P\nu = -\nu/\tau \tag{18}$$

or

$$Q/e = \langle v \rangle = \int_a^b v \, \mathrm{d}^2 r \tag{19}$$

where e denotes electronic charge. Since $v = v_0 \, \mathrm{e}^{-Pt}$ we must calculate

$$\int_a^b v_0 \, \mathrm{e}^{-t/\tau} \, \mathrm{d}^2 r \tag{20}$$

where v_0 is the steady-state distribution for $t < 0$. Since the steady state is here defined by detailed balancing, v_0 is independent of r, a fact which is quickly verified by inspection of equations (7) and (8) with $\partial n_i/\partial t = \partial v/\partial t = 0$. We therefore need only calculate

$$Q \propto \int_a^b \mathrm{e}^{-t/\tau} r \, \mathrm{d}r. \tag{21}$$

Pratt claims to have kept $t \lesssim 2\tau$ while collecting his data (Pratt and Zimmerman 1969), and for this case there exist a number of useful approximations yielding an exponential decay law with an 'effective' lifetime (McCauley 1972). As useful as any is the result of an expansion in moments: by defining

$$\langle \tau \rangle = \frac{4\pi}{b^2 - a^2} \int_a^b \tau r \, \mathrm{d}r \propto 6{\cdot}34 - \ln E_{\mathrm{av}} \tag{22}$$

we can write

$$Q \propto \mathrm{e}^{-t/\langle\tau\rangle} + \sum_{n=2}^{\infty} \left. \frac{\mathrm{d}^n v}{\mathrm{d}\tau^n} \right|_{\langle\tau\rangle} (\tau - \langle\tau\rangle)^n \tag{23}$$

where

$$\langle(\tau - \langle\tau\rangle)^n\rangle = \int_{\tau_a'}^{\tau_b'} (\tau - \langle\tau\rangle)^n \, \mathrm{e}^{2\tau'} \, \mathrm{d}\tau'. \tag{24}$$

τ_a and τ_b are the lifetimes at $r = a$ and $r = b$ respectively. When $t/\langle\tau\rangle = 2$ the second moment vanishes and all higher moments may be neglected so long as $E_{\mathrm{av}} \lesssim 40 \mathrm{~V~cm^{-1}}$, where E_{av} is the 'average field' defined by Pratt (Pratt and Zimmerman 1969). The approximation

$$Q \sim \mathrm{e}^{-t/\langle\tau\rangle} \tag{25}$$

is very good for $E_{\mathrm{av}} \lesssim 40 \mathrm{~V~cm^{-1}}$ but not for $E_{\mathrm{av}} \gtrsim 80 \mathrm{~V~cm^{-1}}$. Only for the former case can one obtain a good fit to the data; this is as it should be since for $E_{\mathrm{av}} \gtrsim 40 \mathrm{~V~cm^{-1}}$ the decay law is not a simple exponential. Clearly the fluid cannot be characterized by a single lifetime whenever E_{av} is so large that both τ_a and τ_b are approximately equal to $\tau_b - \tau_a$. Although Pratt (1967) was aware of this fact, he did not state the limits of applicability of the exponential law.

The comparison of $\langle\tau\rangle$ with Pratt's data is given in figure 4, where it makes no difference at what value of E_{av} one chooses to fit theory to experiment so long as one chooses $5 \lesssim E_{\mathrm{av}} \lesssim 40 \mathrm{~V~cm^{-1}}$ (we have chosen the point $E_{\mathrm{av}} = 10 \mathrm{~V~cm^{-1}}$). The main point of interest is the logarithmic divergence of the lifetime as $E_{\mathrm{av}} \to 0$, mistakenly

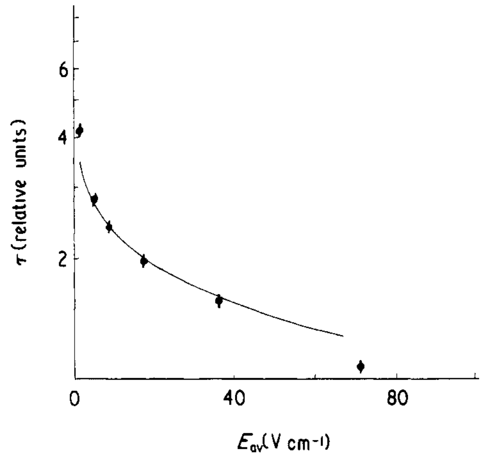

Figure 4. τ plotted against E_{av}(V cm^{-1}) in relative units at T = 1·65 K. Pratt plotted τ divided by its value at 80 V cm^{-1}, while we have chosen to fit our prediction (solid curve) to his data at E_{av} = 10 V cm^{-1}. For $E_{av} \gtrsim$ 50 V cm^{-1} Pratt's data (♦) are slightly inaccurate since the simple exponential decay law assumed by him becomes an increasingly poor approximation with increasing field strength.

attributed by Pratt to 'recapture effects' (Pratt and Zimmermann 1969) due to his use of the formula of Donnelly and Roberts (I) in analysing his data. For $E_{av} \lesssim$ 1 V cm^{-1} there does appear to be some small discrepancy, but this is very likely to be due to the onset of 'initial recombination', a recapture effect involving an attempted escape of an ion from, and subsequent recapture by, a single vortex line. The general theory of this effect has been discussed by Onsager (1938) but it may be explained in simplified terms as follows: with the ion source off ($J_r n_v \sigma = 0$) we have

$$\mathrm{d}v/\mathrm{d}t = -v/\tau + I_R \tag{26}$$

where the additional term I_R represents the current back into the vortex due to recapture of an escaping 'downstream' ion. If $I_R = \alpha' P v$ with $0 \leqslant \alpha < 1$, then we have an effective lifetime

$$\tau_{eff} = \tau/1-\alpha' > \tau \tag{27}$$

whenever α' is independent of v. This effect will become important for weak fields while disappearing for sufficiently strong fields, and should be much enhanced for very weak fields due to the fact that τ is infinite when $E_{av} = 0$.

This concludes our analysis of the experimental data on capture and release (with the exception of the temperature dependence of the trapping lifetime which has been analysed and discussed elsewhere (McCauley 1974)).

5. Summary

Treating the electron bubble as a point particle and assuming a static vortex line, we have succeeded in accounting for the external field dependence of both the capture cross section and the trapping lifetime in the weak-field limit ($E \lesssim$ 70 V cm^{-1}). For $T \gtrsim$ 1·4 K the predicted temperature variation for the cross section appears correct, whereas for

$T \lesssim 1{\cdot}4$ K there exists a discrepancy which remains unsolved. We have also shown that the field dependence of the trapping lifetime is given accurately by our weak-field approximation, noting that the zero-field lifetime is divergent and that the well-known result given by Donnelly and Roberts is inapplicable except in the limit of infinitely large external fields ($E \gg 10^2$ V cm^{-1}). The reason that the field dependence is so accurately predicted upon the basis of the inverse square law potential is that the field dependence is governed by the interaction at a line–bubble separation $q \sim 100$ Å, whereas to predict the temperature dependence of the trapping lifetime one must calculate for the case of the bubble sitting on the (fluctuating) line (McCauley 1974). The apparent divergence between theory and experiment for temperatures $T \lesssim 1{\cdot}4$ K is at present a mystery.

Acknowledgments

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Note added in proof. The question regarding the temperature dependence of the trapping lifetime below 1·4 K has received further attention in recent experiments at Rutgers University (R I Ostermeier and W I Glaberson 1975, preprint).

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